Expectations Hypothesis of the Term Structure of Implied Volatility: Re-examination

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Abstract

Previous studies have tested the expectations hypothesis of the term structure of implied volatility using fixed-interval time-series of at-the-money options. We show, using a stochastic volatility option pricing model, that even the implied volatilities of at-the-money options are not necessarily unbiased and that the fixed-interval time-series can produce misleading results. We then suggest an alternative approach and test the expectations hypothesis using S&P 500 stock index options. Our results do not support the expectations hypothesis: long-term volatilities rise relative to short-term volatilities but the increases are not matched as predicted by the expectations hypothesis. In addition, an increase in the current long-term volatility relative to the current short-term volatility is followed by a subsequent decline.

Expectations Hypothesis of the Term Structure of Implied Volatility: Re-examination

A number of previous studies use the implied volatility in the Black-Scholes (BS, 1973) formula as an estimate of the expected future volatility of the underlying asset over the life of the option.[1] Implied volatility at any moment in time varies with exercise price as well as time to expiration. The pattern of implied volatility across time to expiration is known as the term structure of implied volatility, and the pattern across strike prices is referred to as the volatility smile (Dupire (1994), Derman and Kani(1994), and Rubinstein (1994)) or the volatility sneer (Dumas, Fleming and Whaley (1998)).[2]

This paper pertains to the term structure of implied volatility, empirical results of which are mixed at best despite its economic and practical importance.[3] Poterba and Summers (1986), one of early works on this issue, show that long-term volatilities do not move much in response to shot term volatilities. Stein (1989) derives the relationship between implied volatilities of at-the-money options and times to expiration, using the stochastic volatility model of Hull and White (1987). More specifically, he tests the term structure relationship of the implied volatilities using S&P 100 stock index options, assuming that instantaneous volatility follows a continuous-time mean-reverting AR(1) process. Stein (1989) reports significant evidence of overreaction in the term structure of the stock index options' implied volatilities, given the level of mean reversion in volatility, which is contradictory to the results in Poterba and Summers (1986). The expectations hypothesis maintains that the

[1] Mayhew (1995) and Figlewski (1997) provide reviews on the various uses and misuses of implied volatility.

[2] Dumas, Fleming and Whaley (1998) show that the implied volatilities for S&P 500 index options prior to the October 1987 market crash form a smile pattern, i.e., deep in-the-money and out-of-the-money options have higher implied volatilities than at-the-money options, and after the crash, a sneer pattern appears, i.e., implied volatilities decrease monotonically as the option exercise prices increase relative to the current index level.

[3] For example, volatility forecasting is central for derivatives trading. See also Duffie and Pan (1997) for the importance of the term structure of implied volatility in measuring value-at-risk.

movements in long-term volatility should be consistent with expected future short-term volatilities. However, Stein (1989) shows that the implied volatility of long-term options overreacts to changes in the implied volatility of short-term options, which is contrary to the expectations hypothesis. Campa and Chang (1995) use the model developed by Hull and White (1987) to test the expectations hypothesis. Using foreign currency options, they do not find evidence of such overreaction of long-term volatilities, supporting the expectations hypothesis. Heynen, Kemna and Vorst (1994) examine the overreaction of implied volatilities and show that it varies with the model specification. More specifically, using the European Option Exchange and Amsterdam Stock Exchange options, they show that overreaction disappears if a different process (e.g., EGARCH (1,1) is assumed for the stock return volatility. Further, Diz and Finucane (1993), conducting an alternative set of tests, find no indication of overreaction for S&P 100 stock index options. In summary, the empirical results on the expectations hypothesis of the term structure of implied volatility of options are mixed at best, but with a strong favor of support.

There are several important issues that may contribute to the contradictory results of the previous studies, e.g., misspecifications in the term structure of volatilities, asynchronous transactions (Harvey and Whaley (1991)), difficulties in arbitraging an entire index, the application of the BS European option model to American options (Harvey and Whaley (1991, 1992a, 1992b)), and so on. One particular problem is that estimation errors induced by inaccurate proxies of expected future volatilities may lead to confounded results. This problem is related to the constant volatility assumption of the BS option pricing model. Empirical evidence suggests that stochastic volatility option pricing models perform better than the constant volatility BS model. For example, Bakshi, Cao and Chen (1997) report that a stochastic volatility model typically reduces the BS pricing errors by 25 to 60 percent. A recent study, Das and Sundaram (1999), examine two different volatility models (jump-diffusion and stochastic volatility) to explain observed shapes of the term structure of implied volatilities. Among others, they find that neither model is able to capture all aspects of observed implied volatilities adequately, but that stochastic volatility model performs much better than the other type in producing a great variety of patterns of the term structure of implied volatilities for at-the-money options.[4]

This paper reexamines the expectations hypothesis of the term structure of implied volatility, and is distinguished from previous studies from several important perspectives.

First, all previous studies use at-the-money options under the assumption that implied volatilities of at-the-money options are unbiased.[5] We show that under the assumption of stochastic volatility the implied volatility of even at-the-money options is not necessarily unbiased. This result is consistent with the finding of Longstaff (1994), which is done independently in a different framework. A matrix of implied volatilities across times to expiration and strike prices provides valuable information. Using only one row of the matrix corresponding to at-the-money options often ignores this additional information and can induce systematic biases in the test of the term structure of implied volatility. Second, we show that the common practice of using daily or weekly time series of options to test the expectations hypothesis is faulty and thus can produce misleading results. Third, we propose a new approach to test the expectations hypothesis and apply it to S&P 500 index options.

A correctly specified test of the expectations hypothesis requires that each observation in a series has the same time to expiration so that long-term volatility can be directly compared to short-term volatility. In this paper, rather than using fixed-interval daily or weekly data, we select observations for a given time to expiration to ensure that each observation in a series has the same time to expiration as specified by the expectations hypothesis. We find that long-term volatilities rise relative to short-term volatilities as predicted by the expectations hypothesis, but the increases are not matched as predicted by the expectations hypothesis. In addition, we find evidence that an increase in the current

[4]See also Melino and Turnbull (1990), Heynen, Kemna and Vost (1994) for further evidence. The issue is still admittedly controversial. For instance, Harvey and Whaley (1992b) argue that although the BS constant volatility assumption is violated in practice, the model's predictions are empirically indistinguishable from stochastic volatility option pricing models when the options are at-the-money and have short times to expiration.

[5]Of course, there could be other advantages of using at-the-money options such as high liquidity compared to in-the-money or out-of-the-money options.

long-term volatility relative to the current short-term volatility is followed by a subsequent decline. Also, to compare with previous studies, we conduct the tests of the expectations hypothesis using the implied volatilities of at-the-money options only. We find that at-the-money option implied volatilities adopted by previous studies produce much more favorable results for the expectations hypothesis than the expected volatilities in this paper.

The rest of the paper is organized as follows: In Section I, we provide a brief literature review and point out potential problems in previous studies, focusing on Stein(1989) and Campa and Chang(1995) which are closely related to ours; in particular, problems of using at-the-money options and fixed interval time series data in testing the expectations hypothesis. In Section II, we propose a new approach for testing the expectations hypothesis of the term structure of implied volatility. We then investigate empirically the hypothesis using S&P 500 index options in Section III. Section IV contains concluding remarks.

I. Biases in Testing the Expectations Hypothesis

A. Bias induced by the use of at-the money option implied volatility

This section shows potential bias induced by the use of implied volatility of at-the-money options, focusing on a previous study, Campa and Chang (CC hereafter, 1995). However, the same problem is applied to all previous studies based on at-the-money options.

The BS option pricing model is based on the assumption that the underlying stock price (S) follows a stochastic process with a constant variance ($\bar{V}$). Hull and White (HW, 1987) derive an option pricing formula under the assumption that the underlying asset price process is generated by a stochastic volatility model with uncorrelated risks. Let $C(\bar{V})$ be the BS formula for a European call option, then the HW option pricing formula (f) is related to the BS formula as follows:

$$f(S_t, \sigma_t^2) = \int C(\bar{V}) h(\bar{V} \mid \sigma_t^2) d\bar{V} = E[C(\bar{V})] \tag{1}$$

where h is the conditional density function given the instantaneous volatility, σ_t^2. The mean variance, $\bar{V}$, is defined as

$$\bar{V} = \frac{1}{T-t} \int_t^T \sigma_t^2 dt, \tag{2}$$

where T is the expiration date. The HW model suggests that once the conditional distribution of stochastic volatility is specified, the HW formula is obtained by taking the expectation of the BS formula with respect to volatility. Unfortunately, no analytic form of the conditional density function for stochastic volatility is known.[6]

As shown in the next section, $C(\bar{V})$ is a concave function of $\bar{V}$ for at-the-money options. Therefore, from Jensen's inequality we have $E[C(\bar{V})] \leq C(E[\bar{V}])$. This implies that there exists $0 \leq \theta_t \leq 1$ such that

$$E[C(\bar{V})] = \theta_t C(E[\bar{V}]). \tag{3}$$

Further, for at-the-money options, θ_t is a decreasing function of the time to maturity of the option. This implies that the bias in the implied volatility of at-the-money options will vary with times to maturity.

Assuming that the BS formula is linear in volatility for at-the-money options, CC (1995) derive the following approximate relationship:

$$C(V_{i,j}) = E_i[C(\bar{V}_{i,j})] = (\theta_{j-i})C(E_i[\bar{V}_{i,j}]) = C(\theta_{j-i}^2 E_i[\bar{V}_{i,j}]) \tag{4}$$

where $V_{i,j}$ $(j > i)$ is the implied variance at time i of an at-the- money option with expiration date j, and E_i is the expectation operator taken at time i. Therefore, equation (4) implies:

$$E_i[\bar{V}_{i,j}] = V_{i,j}/\theta_{j-i}^2, \tag{5}$$

which suggests that, under the assumption of a stochastic volatility, the implied variance of at-the-money option is less than the expected variance.

For two equal-time periods 1 and 2, using equations (2) and (5), and the law of iterated expectations, $E_0[\bar{V}_{i,j}] = E_i[\bar{V}_{i,j}] + u_i$, CC (1995) obtain:

$$V_{1,2} = 2(\theta_1/\theta_2)^2 V_{0,2} - V_{0,1} + \varepsilon_1 \tag{6}$$

where $\varepsilon_1 = \theta_1^2 u_1$.

[6] With different sets of assumptions regarding the stochastic process of the volatility and the price of volatility risk, Amin and Ng (1993) and Heston (1993) derive different solutions.

Assuming $\theta_1/\theta_2 = 1$, they then generalize equation (6) for an arbitrary number of periods, k (see equation (8) in CC (1995)) as

$$\sum_{i=0}^{k-1} E_0[\bar{V}_{i,1+i}] = kE_0[\bar{V}_{0,k}] \quad \text{or} \quad \sum_{i=0}^{k-1} E_i[\bar{V}_{i,1+i}] = k(1/\theta_k)^2 V_{0,k} + \sum_{i=1}^{k-1} u_i. \tag{7}$$

Then, the expectations hypothesis implies:

$$\sum_{i=0}^{k-1} V_{i,1+i} = k(\theta_1/\theta_k)^2 V_{0,k} + \sum_{i=1}^{k-1} \varepsilon_i. \tag{8}$$

Using a long-short term variance spread, equation (8) can be rewritten as:[7]

$$(1/k)\sum_{i=0}^{k-1} V_{i,1+i} - V_{0,1} = (\theta_1/\theta_k)^2 V_{0,k} - V_{0,1} + (1/k)\sum_{i=1}^{k-1} \varepsilon_i. \tag{9}$$

A testable implication of the expectations hypothesis can be written then as:

$$(1/k)\sum_{i=1}^{k-1} \{V_{i,1+i} - V_{0,1}\} = \alpha_0 + \beta_0\{(\theta_1/\theta_k)^2 V_{0,k} - V_{0,1}\} + \varepsilon. \tag{10}$$

¿From equation (10), we can see that the implicit assumption in CC (1995) is not $\theta_1/\theta_2 = 1$, but $\theta_1/\theta_k = 1$ for all k, which is a necessary condition for justifying the use of implied volatility of at-the-money options. CC (1995) justify their results in support of the expectations hypothesis by arguing that the direction of the bias in β_0 depends on the covariance between $(V_{0,k} - V_{0,1})$ and $V_{0,k}$. By decomposing the independent variable in equation (10) into $(V_{0,k} - V_{0,1})$ and $\{(\theta_1/\theta_k)^2 - 1\}V_{0,k}$, they postulate that the bias in the estimated slope is similar to the case of an omitted variable, where $\{(\theta_1/\theta_k)^2 - 1\}V_{0,k}$ is the omitted variable (see equation (9) in CC (1995)). Empirically, they find that the sample covariance between $(V_{0,k} - V_{0,1})$ and $V_{0,k}$ is negative for all maturity pairs of options in their sample. CC (1995) claim that by assuming $\theta_1/\theta_2 = 1$, they underestimate the true β_0 and their results are conservative, and thus, relaxing the assumption will strengthen their results in support of the expectations hypothesis.

Contrary to their argument, the effect of the assumption, $\theta_1/\theta_k = 1$, on the parameter estimates in Campa and Chang (1995) may not be negligible. The assumption of $\theta_1/\theta_k =$

[7] Since the level of volatility is characterized by an AR(1) process, following Campa and Chang (1995), we subtract $V_{0,1}$ from both sides of equation (8).

1 may induce a sign change on the estimated slope in equation (10), depending on the magnitude of $(\theta_1/\theta_k)^2$. The negative covariance between $(V_{0,k} - V_{01})$ and $V_{0,k}$ implies

$$\frac{\partial(V_{0,k} - V_{0,1})}{\partial V_{0,k}} < 0, \quad \text{or} \quad \frac{\partial V_{0,1}}{\partial V_{0,k}} > 1 \quad \text{for all } k.$$

By assuming $\theta_1/\theta_k = 1$, we can rewrite equation (10) as:

$$[(1/k)\sum_{i=1}^{k-1}\{V_{i,1+i} - V_{0,1}\} = \alpha_0^* + \beta_0^*(V_{0,k} - V_{0,1}) + \varepsilon. \tag{11}$$

Taking partial derivatives of equations (10) and (11) with respect to $V_{0,k}$ and equating the resulting right hand side expressions give

$$\beta_0\left\{(\theta_1/\theta_k)^2 - \frac{\partial V_{0,1}}{\partial V_{0,k}}\right\} = \beta_0^*\left\{1 - \frac{\partial V_{0,1}}{\partial V_{0,k}}\right\}. \tag{12}$$

By rearranging, we obtain the following relation:

$$\frac{\beta_0^*}{\beta_0} = 1 - \frac{(\theta_1/\theta_k)^2 - 1}{\frac{\partial V_{0,1}}{\partial V_{0,k}} - 1}. \tag{13}$$

Note that $\theta_1/\theta_k > 1$. Therefore, if $(\theta_1/\theta_k)^2 < \frac{\partial V_{0,1}}{\partial V_{0,k}}$, $\beta_0^* < \beta_0$. But if $(\theta_1/\theta_k)^2 > \frac{\partial V_{0,1}}{\partial V_{0,k}}$, β_0^* has an incorrect sign. In addition, the misspecification induced by the use of implied volatility of at-the-money options does not guarantee consistent estimates of the variance, which may lead to inaccurate test statistics. Accordingly, the conclusions of the Wald test in CC (1995) are highly suspect.

B. Biase induced by the use of fixed-interval times series

Stein (1989) assumes that instantaneous volatility, σ_t, evolves according to an Ornstein-Uhlenbeck process:[8]

$$\sigma_t = \alpha(\mu - \sigma_t)dt + \gamma\sigma_t dz_t \tag{14}$$

where α, μ, γ are unknown positive parameters. As shown in Renault and Touzi (1996), by integrating the stochastic differential equation (14), we obtain

$$\sigma_t = \rho^{\triangle t}\sigma_{t-1} + \mu[1 - \rho^{\triangle t}] + \gamma\int_{t-1}^{t}\rho^{\triangle t}dz_s, \tag{15}$$

[8]This stochastic process of volatility is also considered in Heyen, Kemna and Vost (1994), Diz and Finucane (1993), and Renault and Touzi (1996).

where $\rho = e^{-\alpha}$ and $\triangle t$ is the time interval between $t-1$ and t. Therefore, σ_t can be expressed as an AR(1) process:

$$\sigma_t = a\sigma_{t-1} + b + \varepsilon_t, \quad \varepsilon_t \sim iidN(0, w^2), \tag{16}$$

where

$$a = \rho^{\triangle t};$$

$$b = \mu(1-\rho^{\triangle t}); \text{ and}$$

$$w^2 = \gamma^2 \frac{1-\rho^{2\triangle t}}{2\alpha}.$$

Then, the expected volatility at time t of the stock return with time remaining until expiration T can be shown as

$$E_t[\sigma_{t,t+T}] = \mu + \frac{1-\rho^T}{\alpha T}(\sigma_t - \mu). \tag{17}$$

The hypothesis test in Stein (1989) is based on the following relationship:

$$E_t[\sigma_{t,t+T}] - \mu = \frac{T(1-\rho^K)}{K(1-\rho^T)}(E_t[\sigma_{t,t+K}] - \mu), \tag{18}$$

where K is longer time to expiration than T.

Using implied volatilities of at-the-money options for expected volatilities, Stein (1989) rejects the expectations hypothesis. He first estimates ρ using a short-term volatility time series. The estimate of the instantaneous autocorrelation coefficient, ρ, gives the "theoretical value" of the slope in equation (18). This value is then compared with the estimated slope of the regression of a longer-term volatility series on a shorter-term volatility series. Therefore, the conclusion in Stein (1989) depends upon the estimate of ρ, which is subject to the potential bias induced by using implied volatility of at-the-money options.

Stein (1989) uses a weekly short-term (less than a month to expiration) volatility series to estimate the autocorrelation coefficient. Such a practice, however, ignores term structure effects and thus is misleading because the remaining time to expiration varies from week to week.

With a fixed time interval $\triangle t$ between t and $t+1$, we obtain the following relationship for the given expiration date:[9]

$$E_t[\sigma_{t+1,t+T}] - \mu = \frac{T(\rho^{\triangle t} - \rho^T)}{(T-\triangle t)(1-\rho^T)}(E_t[\sigma_{t,t+T}] - \mu). \tag{19}$$

Equation (19) suggests that the coefficient, ρ, changes in response to changes in time as the options approach expiration. It is clear that the autocorrelation, $\rho^{\triangle t}$, will be properly estimated only when the time-to-maturity remains constant over time.

CC (1995) is subject to the same problem. In either Stein (1989) or CC (1995), the relationship specified as a testable form of the expectations hypothesis does not hold as the time changes. For example, if equation (10) holds between three-month (90 days) and six-month (180 days) variances today ($k = 2$), on the next day the relation will be between 89-day and 179-day variances and two months later the relationship will be between one-month and four-month variances ($k = 4$). If the short-term and long-term options have three months and six months to expiration today, respectively, we need another three-month option three months later to test the expectations hypothesis as specified in equation (10). We can observe this relationship every three months; that is, the time interval should be equal to the time to expiration of the shorter-term option.

II. Unbiased Expectation and the Expectations Hypothesis

The degree of curvature of the BS call price with respect to the implied variance can be determined by looking at the second derivative of C as:

$$C''(\bar{V}) = \frac{S\sqrt{T}}{4\bar{V}^{3/2}}n(d_1)(d_1 d_2 - 1) \tag{20}$$

where

$$d_1 = \frac{\log{(S_0/X)} + (r + \bar{V}/2)T}{\sqrt{\bar{V}T}}, \quad d_2 = d_1 - \sqrt{\bar{V}T},$$

[9]From equation (17), we have

$$E_t[\sigma_{t+1,t+T}] - \mu = \frac{(1-\rho^{T-\triangle t})}{\alpha(T-\triangle t)}(E_t[\sigma_{t+1}] - \mu).$$

We also have $E_t[\sigma_{t+1}] = \mu + \rho^{\triangle t}(\sigma_t - \mu)$ by equation (16). Substituting this into equation (18) and using equation (17) results in equation (19).

X = exercise price, r = risk-free interest rate less dividend yield, T = time to maturity, and $n(\cdot)$ = standard normal density function. The curvature of C is determined by the sign of $d_1 d_2 - 1$. The point of inflection in C is then obtained by solving $d_1 d_2 = 1$ for $\bar{V}$; that is,

$$\bar{V} = I \equiv \frac{2}{T}\{\sqrt{1 + [\log(S_0/X) + rT]^2} - 1\}. \tag{21}$$

Thus, C is a convex function of $\bar{V}$ for sufficiently large or small X; that is, $I - \bar{V} > 0$, which implies that the BS formula underprices deep in-the-money or out-of-the-money options. For at-the-money options ($S = X$), however, $I = 0$ and C is always a concave function of $\bar{V}$ and the BS formula overprices them.[10] Also, for at-the-money options, $C''(\bar{V})$ is a decreasing function of T, other things being constant, that is, the magnitude of overpricing becomes larger as the time to expiration increases. This suggests that θ_t, introduced in section I, is a decreasing function of the time to maturity and that the volatility implied by the BS formula even for at-the-money options is biased.

The inflection point occurs at slightly in-the-money or out-of-the-money options for which $\bar{V} = I$. In principle, one can take the implied volatility as an unbiased expectation if an option exists for which the implied variance (V) is exactly equal to I. If there is no option for which $V = I$, then one can take the option with smallest $|I - V|$ among existing options and use the following approximation:

$$E[\bar{V}] = \frac{f - C(I)}{C'(I)} + I \equiv V^e, \tag{22}$$

which is derived from the Taylor series expansion of $E[C(\bar{V})]$ around I, that is,

$$f = E[C(\bar{V})] = C(I) + C'(I)(E[\bar{V}] - I), \tag{23}$$

where $C'(I)$ is the first derivative of $C(\bar{V})$ evaluated at I, where[11]

$$C'(\bar{V}) = Xe^{-rT} n(d_2) \frac{\sqrt{T}}{2\sqrt{\bar{V}}}.$$

[10]It is possible that C is convex for very long-term options. But typical options have maturity less than a year and hence the BS formula is concave for most options.

[11]Note that $C''(I) = 0$ by definition.

Using the expected variance obtained from equation (22), we can specify the relationship in equation (10) as follows:

$$(1/k)\sum_{i=1}^{k-1}(V^e_{i,1+i} - V^e_{0,1}) = \alpha_0 + \beta_0(V^e_{0,k} - V^e_{0,1}) + \varepsilon. \tag{24}$$

Equation (7) can also be used to derive an implication of the expectations hypothesis regarding the predicted change in the long-term variance over a short-term period. Rewriting equation (7) for the long-term variance of an option with time to expiration $(k-1)$ at $i = 1$, we get

$$(k-1)E_1[\bar{V}_{1,k}] = \sum_{i=1}^{k-1} E_1[\bar{V}_{i,1+i}]. \tag{25}$$

By taking the difference between equation (25) and equation (7), we obtain the one-period change in the long-term variance as follows:

$$V^e_{1,k} - V^e_{0,k} = \alpha_1 + \beta_1 \frac{1}{k-1}(V^e_{0,k} - V^e_{0,1}) + \varepsilon. \tag{26}$$

If the expectations hypothesis holds, the changes in the long-term variance should reflect the term structure spread. Thus, the expectations hypothesis implies that the estimated slope coefficient of β_1 is equal to one.

III. Empirical Results

A. Data

In this study, we use the S&P 500 index options traded on the Chicago Board of Options Exchange (CBOE).[12] We have chosen the S&P 500 index options partly because they are one of the most actively traded options exhibiting a large range of exercise prices, which

[12]We performed the same set of tests for the Philadelphia Stock Exchange (PHLX) currency options of British Pound, Canadian Dollar, and Deutsche Mark from 1988 to 1997. Due to thin trading in European currency options, especially for long-term options, we were able to test only for the short-term options up to four months of maturity. The results are qualitatively similar to those reported in Table II and will be available from the authors upon request.

facilitate our comparison of implied volatilities over different exercise prices. Also, they are European options for which the BS model is developed.[13]

We use the average of daily closing bid and ask prices from January 1989 to March 1999. The expiration date is set as the first business day after the third Friday of the contract month, and the last trading day occurs on the business day (usually a Thursday) preceding the third Friday when the exercise-settlement value is calculated. Currently, the exercise-settlement value is calculated using the opening reported sales price of each component stock of the index on the last business day (usually Friday) before the expiration date.[14] The index prices, dividend yields, and interest rates are obtained from Datastream. Datastream provides various daily interest rates of different maturities including overnight, one-week, one-month, three-month, six-month, and one-year. We use linear interpolation to calculate the intermediate interest rates. We adjust for dividend payments by subtracting annualized dividend yields from interest rates when we calculate implied volatility and estimate the expected volatiltiy.

B. Time to Expiration and Exercise Price Effects of Implied Volatility

Figure 1 displays the implied variances of the call options in the sample. The time to maturity ranges from 2 weeks to 40 weeks and the spot to strike price ratio, S/X, from .72 to 1.25. Clearly, for the options with a time to maturity less than 4 weeks, the implied variances show a smile pattern, i.e., the implied variances tend to be greater for deep in-the-money and out-of-the-money options than for at-the-money options. However, for long-term maturity options, the implied variances seem to follow a sneer pattern found in Dumas, Fleming and Whaley (1998)—that is, the implied variances tend to decrease monotonically as the strike price rises relative to the spot index.

[13]Also, the S&P 500 stock index options do not contain the "wildcard" feature of the S&P 100 stock index options that seriously complicates the valuation procedure.

[14]Before August 24, 1992, the exercise-settlement value was calculated using the closing price of Thursday rather than the opening price of Friday. Dumas, Fleming and Whaley (1998) provides a historical background of the S&P 500 index options.

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Please insert Figure 1 about here.

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To see the convexity of the option prices with respect to the ratio, S/X, we plot the deviations of implied variances from the inflection point, $I-V$, for different times to maturity in Figure 2. All of the plots are parabolic curves. $I-V$ tends to be negative (i.e., C is concave) for at-the-money options regardless of their time to maturity. On the other hand, $I-V$ is always positive (i.e., C is convex) for deep in-the-money or out-of-the-money options. Note that negative $I-V$ implies that C is concave and hence, by Jensen's inequality

$$E[C(\bar{V})] = C(V) \leq C(E[\bar{V}]).$$

Since C is an increasing function of V, it follows that $V \leq E[\bar{V}]$. Therefore, implied volatility for at-the-money options is lower than expected volatility. Similarly, the opposite can be shown for deep in-the-money or out-of-the-money options. Thus, the plots in Figure 2 confirm the simulation results in HW that for at-the-money options, implied volatilities are lower than expected volatilities, and that for deep in-the-money and out-of-the-money options, implied volatilities are higher than expected volatilities. Note also that, as the time to expiration increases, the crossing-points (where the implied variance is equal to the expected variance ($I-V=0$)) tend to deviate more from the line of symmetry; that is, the parabola becomes wider. This suggests that the BS option price will approximate a linear function of volatility for deep in-the-money or out-of-the-money options as the time to maturity increases.

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Please insert Figure 2 about here.

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Also, in order to see how implied volatility behaves for different times to maturity, we plot the deviation of implied variance from the inflection point against the time to maturity. Figure 3 presents the results. The deviations for at-the-money options tend to be negative and decrease as the time to expiration increases. For in-the-money options and out-of-the-money options, the deviations are dispersed widely for very short-term maturities, close to zero for mid-term maturities and tend to decrease as the time to maturity increases.

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Please insert Figure 3 about here.

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Figure 4 depicts the deviation of the implied variance from the inflection point across the ratio of the underlying stock index price to the option's strike price and the time to maturity. Note that the inflection point as defined in equation (21) depends upon not only the spot to strike price ratio and the time to maturity but also the interest rate and dividend yield. After different interest rates and dividend yields are accounted for through the inflection point, the figure becomes much smoother than that in Figure 1. The deviation of implied variance from the inflection point is greater for in-the-money and out-of-the money options than at-the-money options. However, the difference does not appear to be significant for long-term maturity options.

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Please insert Figure 4 about here.

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Table I provides details of the average deviations of BS implied volatility (V) from the inflection point (I) for varying ratios of S/X and different times to expiration. The average deviations tend to be bell-shaped over S/X; they are smaller for at-the-money options or near at-the-money options and greater for deep in-the-money and out-of-the-money options.

This table confirms that the deviations tend to decrease for at-the-money options as the time to expiration increases. For in-the-money and out-of-the-e money options, the deviations are positive for short-term maturities, but become negative as the time to maturity increases. For a given maturity, the inflection point (i.e., $I - V = 0$) occurs at slightly in-the-money and out-of-the-money options.

In section I, we have shown that the use of implied volatility of at-the-money options can produce misleading results in testing the expectations hypothesis. The results in Table

I are support our argument, suggesting that the implied volatility of even at-the-money options can be more biased than that of in-the-money or out-of-the-money options for some maturity options. However, the overall effect of the biases induced by the use of implied volatility of at-the-money options in testing the expectations hypothesis cannot be predicted and is open for an empirical investigation, which is the subject in the next section.

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Please insert Table I about here.

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C. Tests of the Expectations Hypothesis

To eliminate the varying time to expiration effect, we choose observations with a fixed time to expiration rather than using daily or weekly data for the same expiration dates. As a result, each observation in a series of the sample has the same time to expiration, and the time interval between observations is the time to expiration of the short-term option. For example, when we consider the relationship of volatilities between 3 months and 1 month, we choose 89 to 92 days for 3 months and 29 to 32 days for 1 month.[15] Thus, we have several overlapping observations during the same time period, which requires a special correction for the standard errors. To correct for a moving average error term and for conditional heteroskedasticity resulting from the overlapping time series, we report Newey-West (1987) standard errors. However, even with Newey-West standard errors, an asymptotic approximation is not available for the overlapping samples. Accordingly, we estimate the standard errors of the regression coefficients by using the Efron (1979) bootstrapping method based on 1,000 runs.[16]

[15]If we denote m_i as the number of days for ith month, the possible combinations for (m_1, m_2, m_3) are (30, 30, 30), (31,30,31), (28,32,30), and so on. Similar rules are used for other pairs of long- and short-term maturity options. This results in several consecutive daily observations for certain months.

[16]In this procedure, each simulation run preserves the original structure of the variable series and draws a random sample of errors from the original regression with replacement, creating new averages of future

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Please insert Table II about here.

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The estimation results for equations (24) and (26) are reported in Table II. We choose implied variance with the smallest value of $|I - V|$ and adjust it using equation (22). To compare with previous studies, we also report the estimation results using at-the-money options' implied variances.

When we use the expected variance as in equation (22), the expectations hypothesis is not supported. We reject the null hypothesis of $\beta = 1$ for all cases. Most of α values are close to zero. In Panel A, the estimates of β are significantly positive but not close to one as they should be under the expectations hypothesis, suggesting that the long-term variances rise relative to the short-term variances, but the increases are not perfectly matched as predicted by the expectations hypothesis.

However, when we use the implied variances of at-the-money options like previous studies, the results are completely different. The coefficient estimates are much larger than those obtained by the approximated expected variances. Based on the results, we cannot reject the null hypothesis of $\alpha = 0$ and $\beta = 1$ for three out of five cases. In Figure 3 and Table I, the implied variances of at-the-money options are significantly different from the inflection point especially for short-term maturity options. Using the implied variances of one-month at-the-money options appears to give overestimates of the slope coefficients in equation (24) due to the strike price bias and time-to-expiration bias of the implied variances of at-the-money options.

In Panel B, the slope coefficient estimates of equation (26), using the expected variances, are shown to be negative. The negative signs of the estimates suggest that the long-short term variance spread predicts the wrong direction in the subsequent change of the long-term volatility. In other words, a rise in the current long-term volatility relative to the

short-term volatilities. We then estimate β and store the results. The whole operation is then repeated for 1,000 bootstrap samples, at the end of which we have 1,000 estimates of β. These estimates are then used to estimate the asymptotic standard error of β.

current short-term volatility is followed by a subsequent decline, rather than a rise in the long-term volatility in the next period. When we estimate the coefficients using the implied variances, we have similar results but the coefficient estimates tend to be greater than those obtained with expected variances. The bootstrap standard errors are similar to the reported Newey-West standard errors.

In summary, our results do not support the expectations hypothesis. Instead, we have found puzzling behavior in the volatility term structure. The movement of average future short-term volatilities is in the direction predicted by the expectations hypothesis, but not the short-run movement of long-term volatilities.[17]

IV. Conclusion

Previous studies have tested the expectations hypothesis of the term structure of implied volatility using fixed-interval time-series of at-the-money options. We show, using a stochastic volatility option pricing model, that the implied volatilities of at-the-money options are not necessarily unbiased and that the fixed interval time-series can produce misleading results. We then propose a new approach to test the expectations hypothesis and apply it empirically to S&P 500 index options. Rather than simply taking implied volatilities of at-the-money options, we derive a measure of expected variances from a range

[17]This parallels the same puzzling phenomenon in the interest rate term structure literature (e.g., Froot (1989) and Hardouvelis (1994)), and certainly deserves further study. Two primary explanations to the puzzle have been proposed. Campbell and Shiller (1991) argue that movements in current long-term rates obey the general direction predicted by the expectations hypothesis, but those movements are sluggish relative to the movements of the current short rates, i.e., long-term rates underreact relative to current short-term rates or overreact relative to future short-term rates. The second explanation assumes that market expectations are rational but that the information in the spread is composite information reflecting both expected future rates and the variation of risk premia. Froot (1989), using U.S. survey data on short-term and long-term interest rates, finds that the negative correlation between changes in long-term rates and the previous long-short spread is not due to a time-varying risk premium, but due to a violation of the rational expectations assumption, namely, an overreaction to the spread. We may apply an analogous argument to the case of the volatility term structure results found in this study.

of option contracts with different strike prices based on the functional relationship between the HW stochastic volatility model and the BS model. To eliminate varying time to maturity effects, we select observations for a given time to expiration in such a way that each observation in a series has the same time to expiration as specified by the expectations hypothesis.

Our results do not support the expectations hypothesis. Even though the movement of average future short-term volatilities follows the direction predicted by the expectations hypothesis, the short-run movement of long-term volatilities does not: long-term volatilities rise relative to short-term volatilities but the increases are not matched as predicted by the expectations hypothesis. In addition, an increase in the current long-term volatility relative to the current short-term volatility is followed by a subsequent decline. The results also suggest that the specifications based on at-the-money options' implied volatilities adopted by previous studies produce much more favorable results for the expectations hypothesis than those based on our expected future volatility measure.

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Table I

Average Deviations ($I - V$) of BS Implied Variance (V) from Inflection point (I) for Varying Values of S/X and Time to Maturity (T)

S represents the stock price and X represents the exercise price. Each number in the first column represents a mid point of .01 interval: e.g., the values of S/X over (.905, .914) is taken as .91, over (.925, .934) as .93, and so on. Numbers in brackets are medians and numbers in parentheses are standard errors.

	T (Weeks)							
S/X	**2**	**4**	**8**	**12**	**16**	**20**	**24**	**30**
0.85	.5548	.2756	.1308	.0759	.0456	.0336	.0235	.0044
	[.5488]	[.2745]	[.1325]	[.0758]	[.0429]	[.0408]	[.0373]	[.0007]
	(.0482)	(.0199)	(.0158)	(.0165)	(.0148)	(.0182)	(.0256)	(.0200)
0.90	.2193	.1058	.0421	.0200	.0187	.0030	–.0079	–.0192
	[.2184]	[.1047]	[.0464]	[.0274]	[.0163]	[.0089]	[.0056]	[–.0200]
	(.0109)	(.0125)	(.0123)	(.0126)	(.0138)	(.0188)	(.0158)	(.0149)
0.95	.0404	.0201	–.0700	–.0137	–.0164	–.0178	–.0216	–.0289
	[.0440]	[.0145]	[–.0004]	[–.0056]	[–.0097]	[–.0106]	[–.0210]	[–.0239]
	(.0124)	(.0158)	(.0147)	(.0158)	(.0169)	(.0167)	(.0130)	(.0126)
0.97	–.0013	–.0136	–.0199	–.0224	–.0262	–.0251	–.0263	–.0338
	[.0026]	[–.0084]	[–.0130]	[–.0193]	[–.0218]	[–.0258]	[–.0287]	[–.0324]
	(.0069)	(.0063)	(.0324)	(.0086)	(.0084)	(.0086)	(.0081)	(.0087)
1.00	.0029	–.0147	–.0240	–.0259	–.0267	–.0305	–.0345	–.0351
	[–.0089]	[–.0137]	[–.0180]	[–.0184]	[–.0204]	[–.0231]	[–.0260]	[–.0289]
	(.0099)	(.0098)	(.0095)	(.0088)	(.0086)	(.0066)	(.0089)	(.0105)
1.05	.0031	–.0159	–.0256	–.0271	–.0259	–.0275	–.0273	–.0367
	[.0053]	[–.0073]	[–.0152]	[–.0161]	[–.0201]	[–.0208]	[–.0227]	[–.0272]
	(.0134)	(.0092)	(.0078)	(.0082)	(.0120)	(.0091)	(.0049)	(.0078)
1.10	.1278	.0483	–.0001	–.0135	–.0197	–.0444	–.0393	–.0351
	[.1329]	[.0571]	[.0101]	[–.0122]	[–.0182]	[–.0408]	[–.0374]	[–.0324]
	(.0118)	(.0069)	(.0084)	(.0099)	(.0087)	(.0093)	(.0107)	(.0114)
1.15	.2862	.1349	.0490	.01862	.0026	–.0146	–.0138	–.0267
	[.3052]	[.1457]	[.0607]	[.0227]	[.0075]	[–.0086]	[–.0090]	[–.0170]
	(.0179)	(.0205)	(.0108)	(.0103)	(.0095)	(.0106)	(.0063)	(.0131)
1.25	.8611	.4005	.1312	.0936	.0559	.0391	.0151	–.0107
	[.8753]	[.4029]	[.1196]	[.1183]	[.0735]	[.0463]	[.0373]	[–.0252]
	(.0419)	(.0137)	(.0287)	(.0179)	(.0126)	(.0134)	(.0231)	(.0260)

Table II
Tests of the Expectations Hypothesis

The column under Implied Variance represents estimation results using implied variances for at-the-money options and the column under Expected Variance represents the results using $V^e \equiv \frac{f-C(I)}{C'(I)}+I$, where f is the option price, $C(\cdot)$ is the BS formula, and I is the inflection point. Equation (24) is $(1/k)\sum_{i=1}^{k-1}(V^e_{i,1+i} - V^e_{0,1}) = \alpha_0 + \beta_0(V^e_{0,k} - V^e_{0,1}) + \varepsilon$, and equation (26) is $V^e_{1,k} - V^e_{0,k} = \alpha_1 + \beta_1\frac{1}{k-1}(V^e_{0,k} - V^e_{0,1}) + \varepsilon$. Numbers in parentheses are Newey-West standard errors and numbers in brackets are asymptotic standard errors based on Monte Carlo bootstrap simulations of 1,000 runs. n indicates the number of observations in each long-short maturity series.

		Panel A. Equation (24)			
Long-Short	n	Implied Variance		Expected Variance	
		α_0	β_0	α_0	β_0
2 months-1 month	127	.0001	.7348	.0003	.3954*
		(.0003)	(.1393)	(.0013)	(.0728)
		[.0003]	[.1342]	[.0013]	[.0728]
3 months-1 month	92	.0014	.8092	.0010	.3803*
		(.0014)	(.1143)	(.0015)	(.0408)
		[.0014]	[.1149]	[.0015]	[.0416]
4 months-1 month	91	.0020	.7802	.0024	.3127*
		(.0017)	(.1506)	(.0017)	(.0554)
		[.0016]	[.1519]	[.0017]	[.0550]
6 months-2 months	47	.0033	.3465*	.0041	.3652*
		(.0012)	(.0723)	(.0012)	(.0656)
		[.0012]	[.0699]	[.0012]	[.0665]
8 months-2 months	27	.0089	.3694*	.0087	.2881*
		(.0023)	(.0905)	(.0024)	(.0708)
		[.0023]	[.0907]	[.0023]	[.0700]
		Panel B. Equation (26)			
		α_1	β_1	α_1	β_1
3 months-1 month	100	–.0017	–.9075*	–.0007	–1.0425*
		(.0024)	(.2397)	(.0025)	(.1435)
		[.0025]	[.2398]	[.0024]	[.1429]
4 months-1 month	109	.0036	–.9598*	.0041	–1.0055*
		(.0020)	(.2963)	(.0019)	(.1761)
		[.0020]	[.2876]	[.0019]	[.1740]
6 months-2 months	58	.0024	–.8602*	.0015	–.8729*
		(.0024)	(.2551)	(.0025)	(.1544)
		[.0025]	[.2561]	[.0025]	[.1655]
8 months-2 months	76	–.0010	–.8054*	–.0001	–.8599*
		(.0018)	(.3172)	(.0019)	(.2179)
		[.0019]	[.3313]	[.0019]	[.2177]

$*$ indicates rejection of the null hypothesis ($\alpha = 0$ and $\beta = 1$) at 5% significance level based on Wald test.

Figure 1. Implied Volatility as a Function of Time to Maturity and Spot / Strike

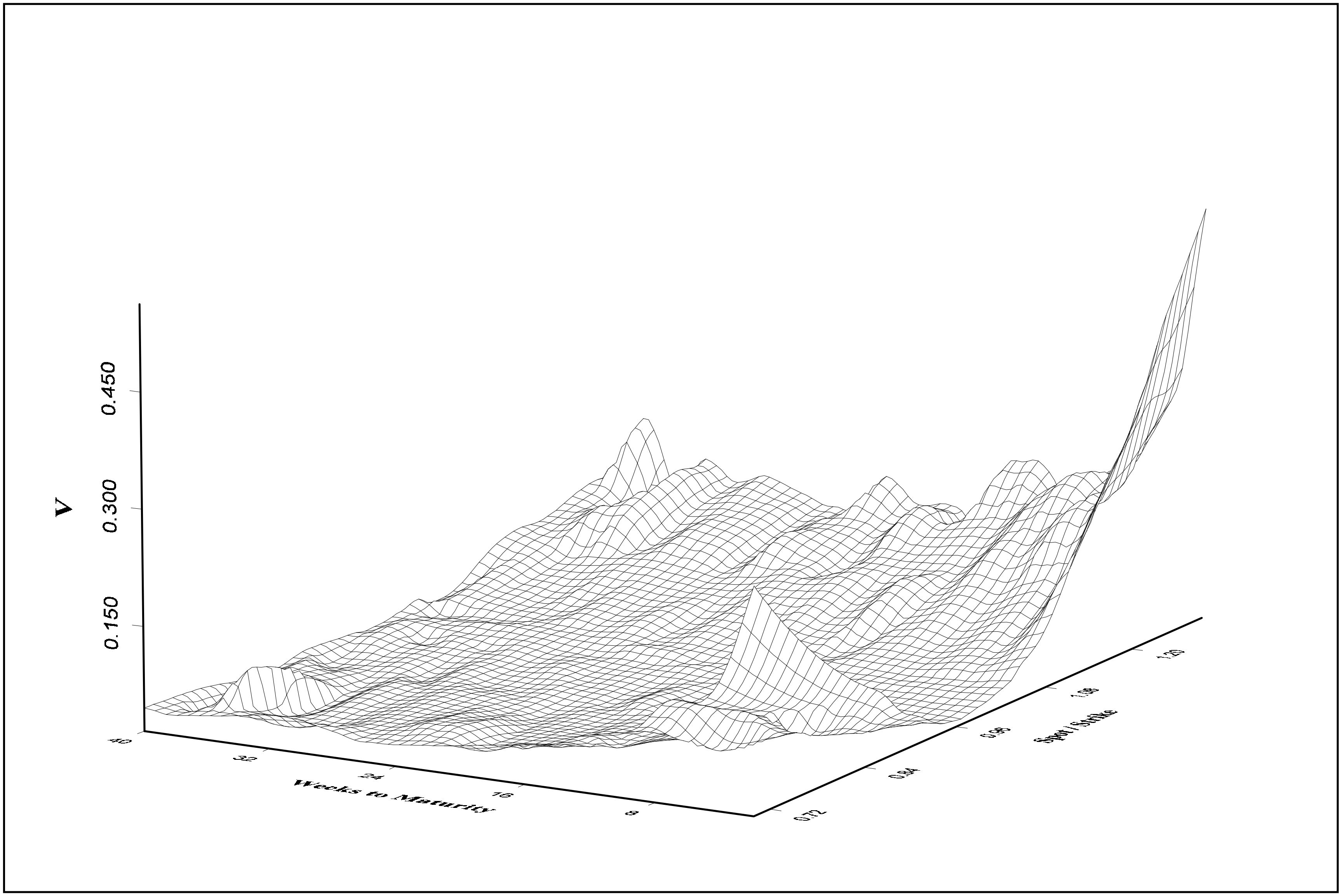

Figure 2. Plots of Deviations of Implied Variance from Inflection Point against Spot / Strike

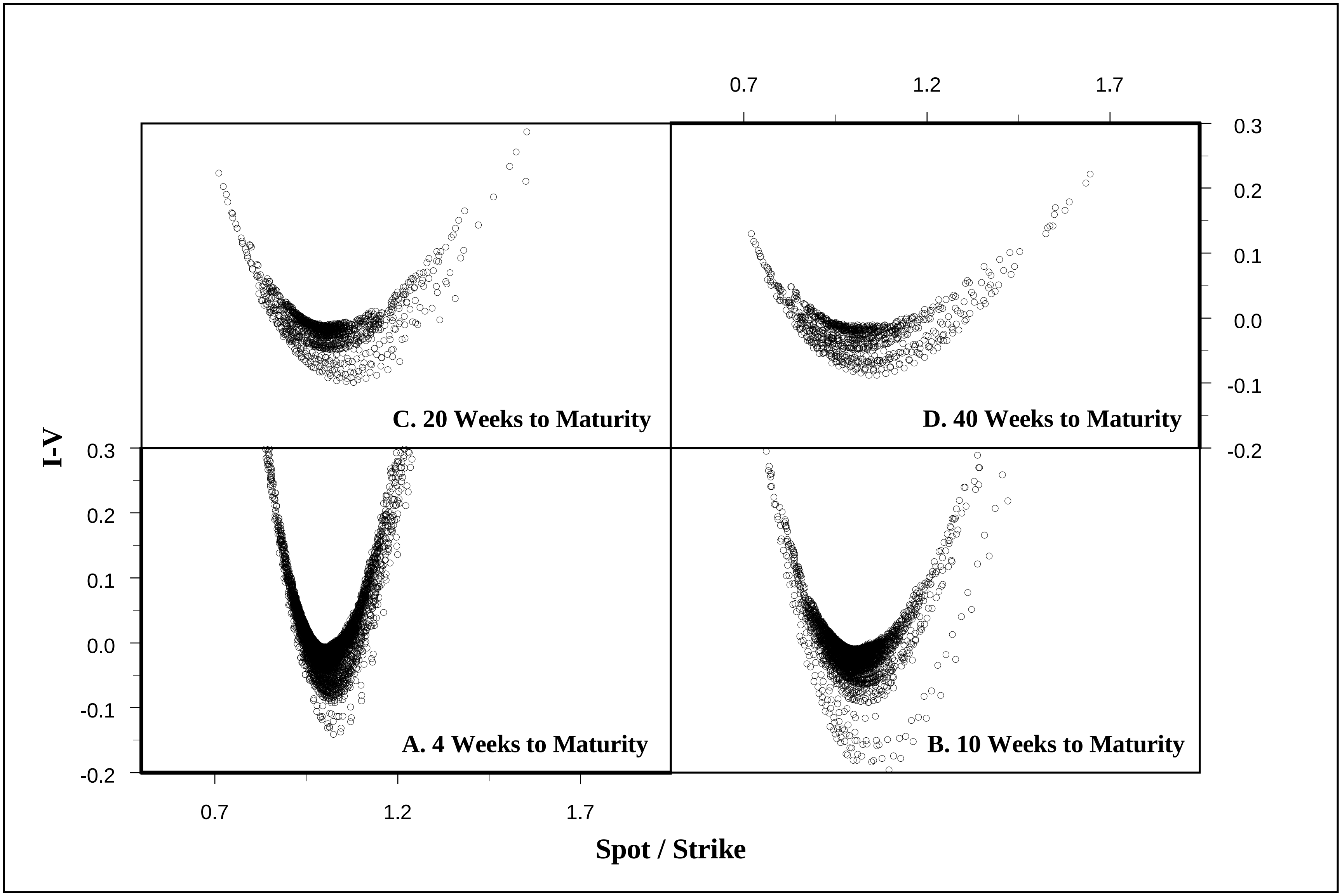

Figure 3. Plots of Deviations of Implied Variance from Inflection Point against Time to Maturity

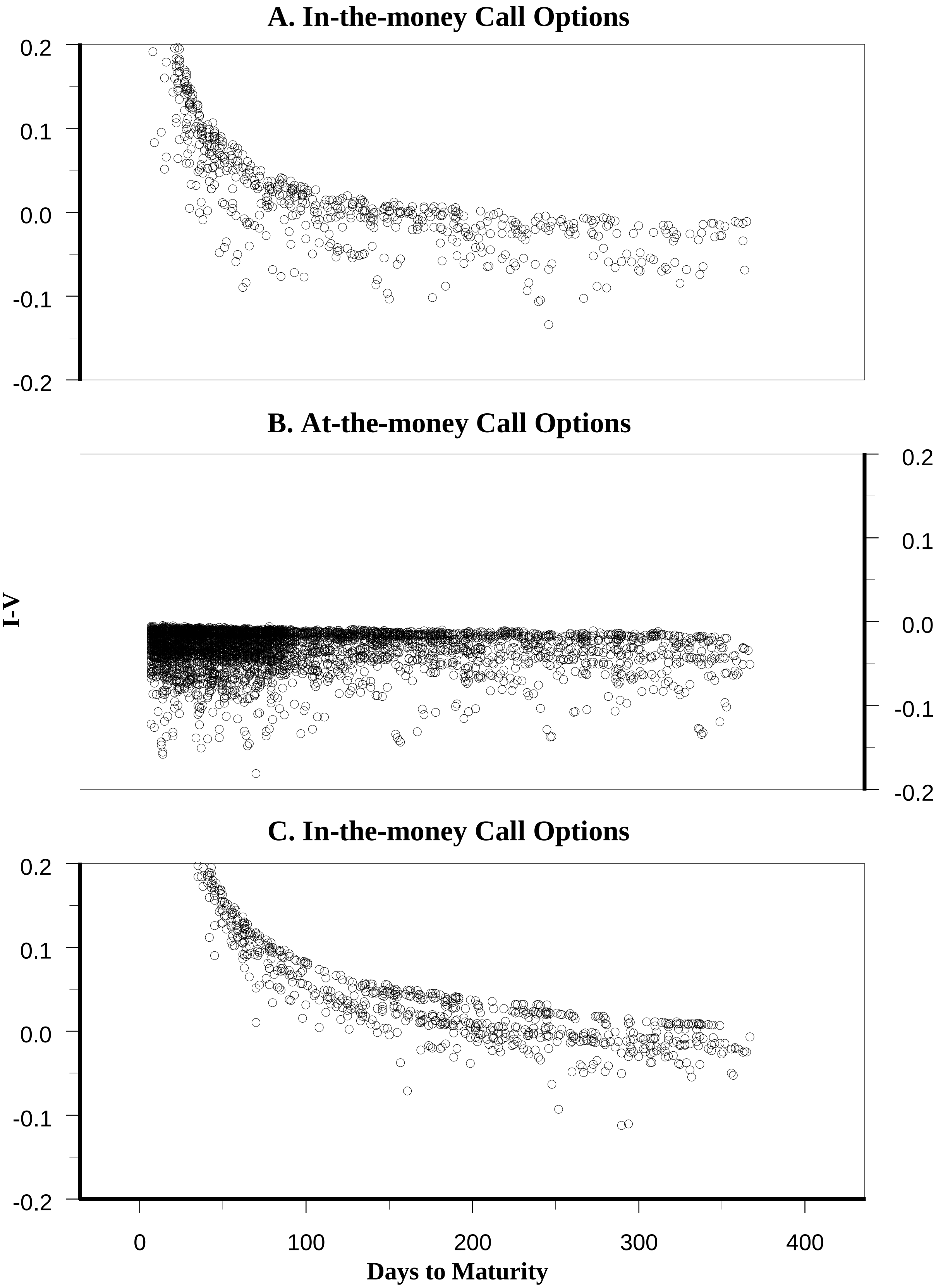

Figure 4. Deviation of Implied Variance from Inflection Point as a Function of Time to Maturity and Spot / Stirke